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ABSTRACT

This paper is concerned with eigenvalue problems for boundary value problems of ordinary differential equations posed on an infinite interval. Problems of that kind occur for example in fluid mechanics when the stability of laminar flows is investigated. Characterizations of eigenvalues and spectral subspaces are given and the convergence of approximating problems which are derived by reducing the infinite interval to a finite but large one and by imposing additional boundary conditions at the far end is proved. Exponential convergence is shown for a large class of problems.

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Key words: Boundary value problems of linear equations, spectral theory of boundary value problems, boundedness of solutions, asymptotic expansion, theoretical approximation of solutions

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SIGNIFICANCE AND EXPLANATION

This paper deals with eigenvalue problems for ordinary differential equations posed on an infinite interval. These problems have the following form: We have a system of linear ordinary differential equations depending linearly on an (eigen) parameter and boundary conditions at a finite point t_0, and we look for nontrivial solutions which fulfill the differential equation on the interval $[t_0,\infty)$ and the boundary condition at t_0, and which have a finite limit at infinity. Such problems occur frequently in fluid mechanics when the stability of flows over infinite media is investigated. In this paper eigenvalues and spectral subspaces are investigated and characterized. A suitable way to solve such problems numerically seems to be to cut the infinite interval at a finite, large enough point T and to impose suitable 'artifical' boundary conditions at $t = T$ in order to get an eigenvalue problem for a two-point boundary value problem on $[t_0,T]$. The questions that arise immediately are: What boundary condition at T achieves convergence of the 'finite' solution to the 'infinite' solution as T converges to infinity, and what order of convergence can we expect. In this paper exponential convergence is shown for well constructed approximating problems.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

EIGENVALUE PROBLEMS ON INFINITE INTERVALS

Peter A. Markowich

1. Introduction

This paper deals with eigenvalue problems of the form

$$y' - t^{\alpha}A(t)y = \lambda t^{\alpha}G(t)y, \quad 1 \leq t < \infty \ , \quad \alpha > -1 \tag{1.1}$$

$$By(1) = 0 \tag{1.2}$$

$$y \in C([1,\infty]) : \Longleftrightarrow y \in C([1,\infty)) \text{ and } \lim_{t\to\infty} y(t) \text{ exists} \tag{1.3}$$

where the $n\times n$ matrices $A, G \in C([1,\infty])$ and $A(\infty) \neq 0$. A sound theory for inhomogenous boundary value problems on infinite intervals has been developed (see Lentini and Keller (1980), de Hoog and Weiss (1980a,b), Markowich (1980a,b,c)) but not much attention has been paid to eigenvalue problems with a singularity of the second kind. de Hoog and Weiss (1976) established a theory for eigenvalue problems in the case that the differential equation has a singularity of the first kind $(\alpha = -1)$ and that $G(\infty) = 0$. They could show that the spectrum has no finite limit point and that the spectral subspaces associated with a particular eigenvalue are finite dimensional. They also considered difference schemes for problems which have been transformed to a finite interval, and they derived convergence results for eigenvalues and spectral subspaces using the collective compactness of the difference schemes. They also derived certain properties of the spectrum and the spectral subspaces of (1.1), (1.2), (1.3) under the assumption that the matrix $A(\infty) + \lambda G(\infty)$ has no eigenvalue on the imaginary axis (see de Hoog and Weiss (1980a)). They showed that all eigenvalues λ for which this assumption on $A(\infty) + \lambda G(\infty)$ holds are isolated and their spectral subspaces are finite dimensional. Their proofs hinge on the Fredholm property of the differential operator.

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The goal of this paper is twofold. First to derive properties of the spectrum and the generalized eigenvectors of (1.1), (1.2), (1.3) and second to consider the approximating eigenvalue problems

$$x_T' - t^\alpha A(t)x_T = \lambda_T t^\alpha G(t)x_T , \quad 1 \le t \le T, \quad T > 1 \tag{1.4}$$

$$Bx_T(1) = 0 \tag{1.5}$$

$$S(T)x_T(T) = 0 . \tag{1.6}$$

These problems, with a suitably chosen matrix $S(T)$ are 'regular' two point boundary eigenvalue problems which can be solved by any appropriate code, for example by collocation (see de Boor and Swartz (1980)). A class of matrices $S(T)$ for which the eigenvalues and spectral subspaces of (1.4), (1.5), (1.6) converge to those of (1.1), (1.2), (1.3) will be defined and the order of convergence, which turns out to be exponential in the most important cases, will be estimated.

This paper is organized as follows. In Chapter two the case where $A(\infty)$ has no eigenvalue on the imaginary axis and where $G(\infty) = 0$ is treated, in Chapter three the assumption $G(\infty) = 0$ is neglected. In Chapter four no assumption on the eigenvalues of $A(\infty)$ are made, but a certain order of convergence of $G(t)$ to 0 is required. In Chapter 5 the Orr-Sommerfeld equation, a fluid dynamical problem posed as an eigenvalue problem on an infinite interval, is dealt with and appropriate approximating problems are devised.

It is of particular interest that the approximation theory in the case $G(\infty) \neq 0$ is treated by using Griegorieff's (1975) 'discrete' approximation theory for eigenvalue problems, which allows the approximating operators to be defined on different spaces which - in some sense - converge to the space on which the eigenvalue problem is posed. This way of pursuing simplfies the analysis essentially.

2. $G(\infty) = 0$: The 'Compact' Case

We assume that $A(\infty) := \lim_{t\to\infty} A(t)$ has no eigenvalue with real part zero and that $G(\infty) := \lim_{t\to\infty} G(t) = 0$. We transform $A(\infty)$ to its Jordan canonical form $J(\infty)$

$$A(\infty) = FJ(\infty)F^{-1} \tag{2.1}$$

and assume that $J(\infty)$ has the blocks structure

$$J(\infty) = \mathrm{diag}(J_\infty^+ , J_\infty^-) \tag{2.2}$$

where J_∞^+ contains all Jordan blocks which have eigenvalues with real part larger than zero and J_∞^- contain all Jordan blocks with eigenvalues with negative real part. Let J_∞^+ be a $r_+ \times r_+$ matrix and J_∞^- a $r_- \times r_-$ matrix and let D_+ resp D_- be the projection onto the sum of invariant subspaces associated with the eigenvalues of J_∞^+ resp J_∞^- . We define a solution operator H of the problem

$$z' = t^\alpha J(\infty)z + t^\alpha g(t) \tag{2.3}$$

for all real $\alpha > -1$ as follows

$$(Hg)(t) = \phi(t) \int_\infty^t D_+\phi^{-1}(s)s^\alpha g(s)ds + \phi(t) \int_\delta^t D_-\phi^{-1}(s)s^\alpha g(s)ds \tag{2.4}$$

where $\delta > 1$ and

$$\phi(t) = \exp\left(\frac{J(\infty)}{\alpha+1} t^{\alpha+1}\right) . \tag{2.5}$$

This operator has been used by de Hoog and Weiss (1980a,b) and they showed that $Hg \in C([1,\infty])$ if $g \in C([1,\infty])$ and that

$$(Hg)(\infty) = -J(\infty)^{-1}g(\infty) \tag{2.6}$$

holds. Moreover $H : C([\delta,\infty]) \to C([\delta,\infty])$ is bounded and its norm is independent of δ and

$$\|(Hg)(t)\| \le \mathrm{const.}\left(\|g\|_{[\frac{t}{2},\infty]} + \left\|\exp\left(\frac{J_\infty^-}{\alpha+1}\left(\frac{2^{\alpha+1}-1}{2^{\alpha+1}}\right)t^{\alpha+1}\right)\right\|\|g\|_{[1,\infty]}\right) \tag{2.7}$$

holds for $t \ge 2$. As norm in $C([\delta,\infty])$ we take $\|\cdot\|_{[\delta,\infty]}$ which denotes the max-norm on the interval $[\delta,\infty]$.

Now we investigate the problem

$$y' - t^\alpha A(t)y = t^\alpha G(t)f(t) \qquad 1 \le t < \infty \tag{2.8}$$

$$By(1) = 0 \tag{2.9}$$

$$(2.10) \qquad y \in C([1,\infty])$$

where $f \in C([1,\infty])$. Substituting $u = F^{-1}y$ the general solution of the transformed problem (2.8),(2.10) is

$$(2.11)(a) \qquad u(t) = \phi(t)G_-\xi + (H(J - J(\infty))u)(t) + (HF^{-1}Gf)(t), \ \xi \in C^{r_-}$$

where $J(t) = F^{-1}A(t)F$ and the $n \times r_-$ -matrix G_- is obtained from D_- by cancelling all columns which have only zero entries.

Obviously the operator

$$(2.11)(b) \qquad H(J - J(\infty)) : C([\delta,\infty]) \to C([\delta,\infty])$$

fulfills $\|H(J - J(\infty))\|_{[\delta,\infty]} < 1$ for δ sufficiently large. Therefore u in (2.11) is defined uniquely on $[\delta,\infty]$ and can be extended uniquely to $[1,\infty]$.

Setting

$$(2.12) \qquad \psi_-(t) = ((I - H(J - J(\infty))^{-1}\phi G_-)(t)$$

$$(2.13) \qquad \psi(Gf)(t) = ((I - H(J - J(\infty)))^{-1}HF^{-1}Gf)(t)$$

we get the general solution of (2.8), (2.9), (2.10):

$$(2.14) \qquad y(t) = F\psi_-(t)\xi + F\psi(Gf)(t), \quad t \in [1,\infty], \ \xi \in C^{r_-} .$$

So (2.8), (2.9), (2.10) is uniquely soluble for every $f \in C([1,\infty])$ if and only if the $r_- \times r_-$ matrix

$$(2.15) \qquad BF\psi_-(1) \ \text{ is regular.}$$

B is assumed to be a $r_- \times n$ matrix. (2.6) and (2.11)(a) imply that $y(\infty) = 0$.

Now we define the operator V as follows:

$$(2.16) \qquad V : \begin{cases} C([1,\infty]) \to C([1,\infty]) \\ f \quad \to Vf = y \end{cases}$$

where y is the solution of (2.8), (2.9), (2.10). V is defined properly if and only if (2.15) holds. This is no restriction because if $\lambda = 0$ is an eigenvalue of (1.1), (1.2), (1.3) we substitute $\lambda = \bar{\lambda} + \gamma$ so that the problem with $A(t)$ replaced by $A(t) + \bar{\lambda}G(t)$ has not $\gamma = 0$ as eigenvalue.

(2.7) and (2.14) imply that V is bounded.

Obviously the eigenvalue problem (1.1), (1.2), (1.3) is equivalent to

$$Vf = \mu f \tag{2.17}$$

with

$$\mu = \frac{1}{\lambda} . \tag{2.18}$$

Our goal is to show that V is compact. We need the following

__Lemma 2.1:__ Assume $\sigma \in C([\delta,\infty])$, $\sigma(t) \to 0$ as $t \to \infty$, $\sigma > 0$ and $\delta > 1$. then the set A_σ defined by

$$A_\sigma := \{f \in C([1,\infty]) \mid \|f\|_{[1,\infty]} < C_1,\ \|f(t)\| < C_2\sigma(t) \text{ for } t > \delta,\ \|f'(t)\| < C_3 t^\alpha\},$$

for $\alpha \in R$ is conditionally compact in $C([1,\infty])$.

Proof. Given $\varepsilon > 0$ we choose $T = T(\varepsilon) > \delta$ so large that $\sigma(t) < \frac{\varepsilon}{C_2}$ for $t > T$. Obviously there is a finite collection of intervals I_i for $i = 1(1)(N(\varepsilon) - 1)$, whose conjunction is $[1,T]$, and there are points t_i in I_i so that

$$\sup_{f \in A_\sigma} \sup_{t \in I_i} \|f(t_i) - f(t)\| < \varepsilon , \quad i = 1(1)(N(\varepsilon) - 1) .$$

This is fulfilled if $|I_i| < \varepsilon/(C_3 T^\alpha)$ with t_i arbitrary in I_i .

Setting $t_{N(\varepsilon)} = \infty$ Theorem 5 in Dunford and Schwartz (1957) is fulfilled and the Lemma follows.

From (2.11) and (2.7) we conclude that

$$\|(Vf)(t)\| < \text{const}(\|\phi(t)G_-\| + \|J(t) - J(\infty)\|_{[\frac{t}{2},\infty]} + \|G(t)\|_{[\frac{t}{2},\infty]} + \tag{2.19}$$
$$+ \|\exp(\frac{J_\infty^-}{\alpha+1}(\frac{2^{\alpha+1} - 1}{2^{\alpha+1}})t^{\alpha+1})\|)\|f\|_{[1,\infty]} , \quad t > \delta > 2$$

holds because ξ in (2.11)(a) equals $-(BE\psi_-(1))^{-1}E\psi(f)(1)$. Setting

$$\sigma(t) = \|\phi(t)G_-\| + \|J(t) - J(\infty)\|_{[\frac{t}{2},\infty]} + \|G(t)\|_{[\frac{t}{2},\infty]} + \|\exp(\frac{J_\infty^-}{\alpha+1}(\frac{2^{\alpha+1} - 1}{2^{\alpha+1}})t^{\alpha+1})\| \tag{2.20}$$

we notice that $\sigma(t) \downarrow 0$ as $t \to \infty$ and therefore

$$\{Vf \mid f \in C([1,\infty]),\ \|f\|_{[1,\infty]} < 1\} \in A_\sigma \tag{2.21}$$

for some constants C_1, C_2, C_3. So V is a compact operator on $C([1,\infty])$.

Because $C([1,\infty])$ is infinite dimensional and V is bounded the spectrum $\sigma(V)$ consists of an infinite sequence of eigenvalues $\mu_i \neq 0$ of finite algebraic multiplicities and $\mu = 0 \in \sigma(V)$ is the only accumulation point of the μ_i's. This implies that the eigenvalues $\lambda_i = \frac{1}{\mu_i}$ of (1.1), (1.2), (1.3) have no finite accumulation point, they fulfill

(2.22) $$|\lambda_i| \to \infty \quad \text{as} \quad i \to \infty .$$

The spectrum of compact operators is described in Dunford and Schwartz (1957), Chapter VII, Theorem 5.

Let $\mu \neq 0$ be a fixed eigenvalue of V. We want to investigate the spectral subspace associated with μ. The spectral projection is given by

(2.23) $$E = E(\mu) = \frac{1}{2\pi i} \int_\Gamma (z - V)^{-1} dz : C([1,\infty] \to C([1,\infty])$$

where Γ is a circle centered at μ which contains no other eigenvalue of V. Moreover

(2.24) $$\operatorname{rank}(E(\mu)) = m$$

where m is the algebraic multiplicity of μ. Let

(2.25) $$\operatorname{Range}(E) = \operatorname{span}\{\varphi_1, \cdots, \varphi_m\} = N((\mu - V)^\beta)$$

hold, where the φ_i are generalized eigenfunctions of V. N denotes the null space and β the ascent of $\mu - V$.

As the range of E is invariant under V we get

(2.26) $$\varphi_i' - t^\alpha A(t)\varphi_i = \sum_{j=1}^m a_{ij} t^\alpha G(t)\varphi_j, \; B\varphi_i(1) = 0 , \quad \varphi_i \in C([1,\infty]) .$$

The $m \times m$ matrix (a_{ij}) can be assumed to be in Jordan canonical form with the only eigenvalue $\lambda = \frac{1}{\mu}$. This can always be achieved by a basis transformation. So every element φ_k is contained in a finite chain $\varphi_{r_1}, \cdots, \varphi_{r_\ell}$ which fulfills

(2.27) $$\varphi_{r_1}' - t^\alpha(A(t) + \lambda G(t))\varphi_{r_1} = 0 , \quad B\varphi_{r_1}(1) = 0 , \quad \varphi_{r_1} \in C([1,\infty])$$

(2.28) $$\varphi_{r_2}' - t^\alpha(A(t) + \lambda G(t))\varphi_{r_2} = t^\alpha G(t)\varphi_{r_1} , \; B\varphi_{r_2}(1) = 0 , \quad \varphi_{r_2} \in C([1,\infty])$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

(2.29) $$\varphi_{r_\ell}' - t^\alpha(A(t) + \lambda G(t))\varphi_{r_\ell} = t^\alpha G(t)\varphi_{r_{\ell-1}} , \; B\varphi_{r_\ell}(1) = 0, \; \varphi_{r_\ell} \in C([1,\infty]) .$$

Obviously every φ_i fulfills

$$\|\varphi_i(t)\| \leq C_i\sigma(t) \tag{2.30}$$

but a stronger estimate will be advantageous for the order of convergence of the approximating eigenvalue problems (1.4), (1.5), (1.6). Therefore we assume that

$$A\left(\frac{1}{\cdot}\right),\ G\left(\frac{1}{\cdot}\right) \in C^{(\alpha+1)k_-+1}\left([0,\tfrac{1}{\delta}]\right),\quad \delta > 1,\quad \alpha \in \mathbb{N}_0 \tag{2.31}$$

holds, where k_- is the largest algebraic multiplicity of an eigenvalue of $A(\infty)$ with negative real part. Under this assumption it follows from Markowich (1980b) that φ_{r_1} which is the solution of the homogenous problem (2.27) as well as the φ_{r_i} for $i > 1$ which are solutions of inhomogenous problems decay exponentially and they fulfill the estimate

$$\|\varphi_{r_i}(t)\| \leq C_i \exp\left(\frac{(\nu_- + \varepsilon)}{\alpha+1} t^{\alpha+1}\right),\qquad t \geq \delta \tag{2.32}$$

where ν_- is the largest negative real part of eigenvalues of $A(\infty)$ and $\varepsilon = \varepsilon(\delta) > 0$ fulfills $\nu_- + \varepsilon < 0$ and $\varepsilon(\delta) \to 0$ as $\delta \to \infty$. Therefore all elements in $\mathrm{Range}(E(\mu))$ fulfill the estimate (2.32).

Now we want to investigate the convergence of the eigenvalue and generalized eigenvectors of the approximating problems (1.4), (1.5), (1.6). As a notion of the distance of closed subspaces we use the 'gap' (see Osborn (1975)) which is defined as follows

$$\mathrm{gap}(M,N) = \max\Big(\sup_{\substack{x \in M\\ \|x\|=1}} \mathrm{dist}(x,N),\ \sup_{\substack{y \in N\\ \|y\|=1}} \mathrm{dist}(M,y)\Big) \tag{2.33}$$

where M,N are closed subspaces of a Banach space $(X,\|\cdot\|)$ and dist is defined as

$$\mathrm{dist}(x,N) = \inf_{y\in N} \|x-y\| . \tag{2.34}$$

We define the operators V_T for T sufficiently large by

$$V_T : \begin{cases} C([1,\infty]) \to C([1,\infty]) \\ \quad f \quad\ \to V_T f = x_T \end{cases} \tag{2.35}$$

where x_T fulfills

(2.36) $$x_T' - t^\alpha A(t)x_T = t^\alpha G(t)f(t), \quad 1 \leq t \leq T$$

(2.37) $$Bx_T(1) = 0$$

(2.38) $$S(T)x_T(T) = 0$$

and

(2.39) $$x_T(t) = x_T(T) \quad \text{for} \quad t \geq T .$$

This definition makes sense if and only if (2.36), (2.37), (2.38) is soluble for every $f \in C([1,\infty])$ and T sufficiently large. de Hoog and Weiss (1980b) have shown that this is the case if (2.15) holds and the $r_- \times n$ -matrix $S(T)$ fulfills

(2.40) $$\|S(T)\| \leq \text{const.} \quad \text{as} \quad T \to \infty$$

(2.41) $$\|(S(T)FG_+)^{-1}\| \leq \text{const} \quad \text{as} \quad T \to \infty$$

where the $n \times r_+$ matrix G_+ is obtained by cancelling all columns of D_+ which have only zero entries. Moreover the stability estimate

(2.42) $$\|x_T\|_{[1,T]} \leq \text{const.}(\|f\|_{[1,T]} + \|\gamma(T)\|)$$

holds for problems of the form (2.36), (2.37) and

(2.43) $$S(T)x_T(T) = \gamma(T)$$

instead of the homogenous boundary condition (2.38). de Hoog and Weiss (1980b) have also shown that (2.41) is necessary if (2.40) holds and they constructed matrices $S(T)$ fulfilling (2.40), (2.41) more explicitly. Obviously the estimate (2.42) with $\gamma(T) \equiv 0$ and the definition of V_T imply

(2.44) $$\|V_T\|_{[1,\infty]} \leq \text{const.}$$

Every operator V_T is compact because

(2.45) $$\|(V_T f)'\|_{[1,T]} \leq \text{const.}\ T^\alpha \|f\|_{[1,T]}$$

holds and $V_T f$ is constant on $[T,\infty]$. By adding the identity $S(T)y(T) = S(T)y(T)$ to (2.8), (2.9), (2.10) and by subtracting from (2.36), (2.37), (2.38) we get the problem

(2.46) $$(x_T - y)' - t^\alpha A(t)(x_T - y) = 0 , \quad 1 \leq t \leq T$$

(2.47) $$B(x_T - y)(1) = 0$$

(2.48) $$S(T)(x_T - y)(T) = -S(T)y(T) .$$

Applying estimate (2.42) implies

(2.49) $$\|V_T f - Vf\|_{[1,T]} \leq \text{const.}\|y(T)\| \leq \text{const}\ \sigma(T)\|f\|_{[1,\infty]}$$

where σ is defined in (2.20). Also we get

(2.50) $$\|V_T f - Vf\|_{[1,\infty]} \leq 2\|V_T f - Vf\|_{[1,T]} + 2\|y\|_{[T,\infty]}$$

because of (2.39). Therefore V_T converges to V in the norm and

(2.51) $$\|V_T - V\|_{[1,\infty]} \leq \text{const.}\sigma(T)$$

holds.

It should be noticed that $G(t) \to 0$ as $t \to \infty$ is absolutely crucial for the norm convergence.

The *eigenvalue problem* (1.4), (1.5), (1.6) is equivalent to

(2.52) $$V_T f = \mu_T f$$

with

(2.53) $$\mu_T = \frac{1}{\lambda_T} .$$

The generalized eigenfunction of (1.4), (1.5), (1.6) are obtained by restricting the generalized eigenfunction of (2.52) to $[1,T]$.

Because of the compactness of V_T *there is an infinite sequence of eigenvalues* $\mu_T^{(i)} \neq 0$ accumulating at 0. The compactness and norm convergence (2.51) allows us to apply Osborn's (1975) result. We derive the following convergence statements.

For T sufficiently large there are exactly m eigenvalues of V_T, counted according to algebraic multiplicities, which lie in the circle Γ defined in (2.23). These m eigenvalues $\mu_T^1,\cdots,\mu_T^m$ converge to μ as $T \to \infty$. The spectral projections

(2.54) $$E_T = \frac{1}{2\pi i} \int_\Gamma (z - V_T)^{-1} dz$$

fulfill

(2.55) (a) $\text{rank}(E_T) = \text{rank}(E) = m$, (b) $\text{gap}(E_T, E) \to 0$ as $T \to \infty$.

Setting

(2.56) $$\hat{\mu}_T = \frac{1}{m} \sum_{i=1}^{m} \mu_T^i$$

we get

(2.57) $$\max(|\hat{\mu}_T - \mu|, \text{gap}(E_T, E)) \leq \text{const.}\|(V_T - V)|_{\text{Range}(E)}\|_{[1,\infty]}$$

where $(V_T - V)|_{\text{Range}(E)}$ denotes the restriction of $V_T - V$ to Range(E).

Resubstituting $\lambda = \frac{1}{\mu}$, $\lambda_T = \frac{1}{\mu_T}$ and assuming that (2.31) holds we get the following estimate which implies exponential convergence:

$$\max(|\lambda - \frac{1}{\mu_T}|, \ \text{gap}\ (E_T, E)) \leq \text{const.}\ \exp\left(\frac{(\nu_- + \varepsilon)}{\alpha+1} T^{\alpha+1}\right) . \tag{2.58}$$

The estimate for the order of convergence of each λ_T^i to λ is worse:

$$|\lambda - \lambda_T^i| \leq \text{const.}\ \exp\left(\frac{(\nu_- + \varepsilon)}{(\alpha+1)\beta} T^{\alpha+1}\right) , \qquad i = 1(1)m . \tag{2.59}$$

Here β is again the ascent of $\mu - V$. The constants in (2.58), (2.59) are independent of T but may very well depend on λ. If the assumption (2.31) is dropped it is easily derived from Markowich (1980b), Chapter 2 that the generalized eigenfunction decay faster than every (negative) power of t and so the same is true for the order of convergence (2.58), (2.59).

A possible choice for $S(T)$ is

$$S(T) \equiv S \equiv (G_+)^T F^{-1} \tag{2.60}$$

where the superscript T denotes transposition. The condition (2.41) is fulfilled because

$$SFG_+ = I_{r_+} \tag{2.61}$$

holds for the choice (2.60), which has been used by de Hoog and Weiss (1980b) for the solution of inhomogenous boundary value problems on infinite intervals.

3. The Case $G(\infty) \neq 0$.

Again we consider the problem (1.1), (1.2), (1.3) but we drop the restriction $G(\infty) = 0$. We again assume that B is a $r_- \times n$-matrix.

The following assumption will be needed:

(I) The problem

$$y_h' - t^\alpha A(t) y_h = 0 \qquad 1 \leq t < \infty \tag{3.1}$$

$$B y_h(1) = 0 \tag{3.2}$$

$$y_h \in C([1,\infty]) \tag{3.3}$$

has the unique solution $y_h = 0$. This guarantees that the $r_- \times r_-$ matrix

$$B F \psi_-(1) \text{ is regular} \tag{3.4}$$

and therefore the inhomogenous problem

$$y' - t^\alpha A(t) y = t^\alpha G(t) f(t), \qquad 1 \leq t < \infty \tag{3.5}$$

with the boundary conditions (3.2) and (3.3) has a unique solution for every $f \in C([1,\infty])$. Moreover we restrict the eigenparameter λ to an open and connected set $\Omega \subset \mathbf{C}$ with $0 \in \Omega$ so that the matrix $A(\infty) + \lambda G(\infty)$ for $\lambda \in \Omega$ has no eigenvalue $\nu(\lambda)$ on the imaginary axis and therefore the matrices G_+ and G_- are constant for $\lambda \in \Omega$.

de Hoog and Weiss (1980a) proved that all eigenvalues λ of (1.1), (1.2), (1.3) which fulfill $\lambda \in \Omega$ are isolated and that the associated spectral subspaces are finite dimensional. Each (generalized) eigenfunction y associated with an eigenvalue $\lambda \in \Omega$ fulfills $y(\infty) = 0$. The spectrum of (1.1), (1.2), (1.3) has no finite limit point in Ω .

Of course this settles the case $G(\infty) = 0$ completely because then $\Omega = C$ holds but the compactness arguments in Chapter 2 were included because they will be used in Chapter 4 where imaginary eigenvalues of $A(\infty)$ will be admitted.

We define the operator V slightly different to Chapter 2:

$$(3.6) \qquad V : \begin{cases} C_0([1,\infty]) \to C_0([1,\infty]) \\ \\ f \quad \to Vf = y \end{cases}$$

where $C_0([1,\infty])$ is the Banach space of all function $f \in C([1,\infty])$ which fulfill $f(\infty) = 0$ and y is the solution of the problem (2.8), (2.9), (2.10). Assumption (I) makes V well-defined on $C_0([1,\infty])$ and (2.6), (2.11) guarantee that $y(\infty) = 0$ if $f(\infty) = 0$.

The eigenvalue problem (1.1), (1.2), (1.3) is equivalent to

$$(3.7) \qquad Vf = \mu f , \quad f \in C_0([1,\infty])$$

with

$$(3.8) \qquad \mu = \frac{1}{\lambda} , \quad \lambda \in \Omega$$

because all generalized eigenfunction associated with $\lambda \in \Omega$ are in $C_0([1,\infty])$ and because $\lambda = 0$ is no eigenvalue.

Now let us consider a fixed eigenvalue $\mu = \frac{1}{\lambda}$, $\lambda \in \Omega$ with algebraic multiplicity m and acent β. The spectral projection is again given by

$$(3.9) \qquad E = E(\mu) = \frac{1}{2\pi i} \int_\Gamma (z - V)^{-1} dz : C_0([1,\infty]) \to C_0([1,\infty])$$

where the circle Γ centered at μ contains no other eigenvalue of (3.7) and the image of Γ under the mapping $\lambda = \frac{1}{\mu}$ denoted by $\frac{1}{\mu}(\Gamma)$ is in Ω. E fulfills (2.24), (2.25).

We want to approximate the generalized eigenpair $(\lambda, \text{Range } (E(\mu)))$ by a sequence of nearby eigenpairs of (1.4), (1.5), (1.6).

Therefore we define the operators V_T for T sufficiently large

$$(3.10) \qquad V_T : \begin{cases} C([1,T]) \to C([1,T]) \\ \\ f_T \quad \to V_T f_T = x_T \end{cases}$$

where x_T solves (2.36), (2.37), (2.38), $S(T)$ is independent of λ and fulfills (2.40), (2.41). So the V_T's are defined properly and fulfill

$$\|V_T f_T\|_{[1,T]} \leq \text{const.} \|f_T\|_{[1,T]} \tag{3.11}$$

$$\|(V_T f_T)'\|_{[1,T]} \leq \text{const.} T^{\alpha} \|f_T\|_{[1,T]} \ . \tag{3.12}$$

So each V_T is compact and has an infinite sequence of eigenvalues μ_T^i which have the only accumulation point 0 . The associated spectral subspaces are finite dimensional.

It is therefore clear that the finite interval problems (1.4), (1.5), (1.6) can not be used to approximate continuous parts of the spectrum of (1.1), (1.2), (1.3) which may very well exist outside of Ω .

We define the restriction operator

$$r_T : \begin{cases} C_0([1,\infty]) \to C([1,T]) \\ \qquad f \to r_T f = f|_{[1,T]} \end{cases} \tag{3.13}$$

Then for every sequence $T_n \to \infty$ the sequence of spaces $C([1,T_n])$ form a discrete approximation $A(C_0([1,\infty]), \prod_n C([1,T_n]), r_{T_n})$ for the space $C_0([1,\infty])$ in the sense of Stummel (1970).

A sequence $f_{T_n} \in C([1,T])$ is said to converge to an element $f \in C_0([1,\infty])$, denoted by $f_{T_n} \to f$, if

$$\|f_{T_n} - r_{T_n} f\|_{[1,T_n]} \to 0 \quad \text{as} \quad n \to \infty \ . \tag{3.14}$$

A sequence of bounded operators in $C([1,T_n])$, is said to converge to a bounded operator on $C_0([1,\infty])$, again denoted by $A_{T_n} \to A$, if for every $f \in C_0([1,\infty])$ and for every sequence f_{T_n}

$$f_{T_n} \to f \ \text{implies} \ A_{T_n} f_{T_n} \to Af \ . \tag{3.15}$$

We will drop the subscript n mostly.

Taking a fixed $z \neq 0$ in the resolvent set of V and in $\frac{1}{\lambda}(\Omega)$ we investigate $(z - V_T)^{-1}$. Setting $u_T = (z - V_T)^{-1} f_T$ for an arbitrary $f_T \in C([1,T])$ we easily find that

(3.16) $$u_T = \frac{1}{z}(y_T + f_T)$$

where Y_T solves

(3.17) $$y'_T - t^\alpha(A(t) + \frac{1}{z}G(t))y_T = \frac{1}{z}t^\alpha G(t)f_T$$

(3.18) $$By_T(1) = 0$$

(3.19) $$S(T)y_T(T) = 0 .$$

Defining $F(\frac{1}{z})$ as the matrix which transforms $A(\infty) + \frac{1}{z}G(\infty)$ to its Jordan canonical form we derive from de Hoog and Weiss (1980b) that (3.17), (3.18), (3.19) is uniquely soluble for T sufficiently large if

(3.20) $$\|(S(T)F(\frac{1}{z})G_+)^{-1}\| \le \text{const} \quad \text{as} \quad T \to \infty$$

and the estimate

(3.21) $$\|(z - V_T)^{-1}\|_{[1,T]} \le \text{const}(z)$$

follows if (3.20) holds. This bound is uniform in $z \in K_1$, where K_1 is compact, $0 \notin K_1$ and $\frac{1}{\mu}(K_1) \subset \Omega$ (see Kreiss (1972)).

This analysis also shows that

(3.22) $$(z - V_T)^{-1} \to (z - V)^{-1}$$

uniformly for $z \in K_1$. Therefore (3.20) guarantees that

(3.23) $$\inf_\mu |\mu - \mu_T| \to 0 \quad \text{for} \quad t \to \infty$$

where $\mu \in \frac{1}{\lambda}(\Omega)$ are the eigenvalue of V and μ_T are the eigenvalues of V_T. Moreover the spectral projections fulfill

(3.24) $$E_T(\mu) = \frac{1}{2\pi i}\int_\Gamma (z - V_T)^{-1}dz \to E(\mu)$$

(3.25) $$\underline{\lim}_{T\to\infty} \text{rank}(E_T(\mu)) \ge \text{rank}(E(\mu)).$$

The sets $\text{Range}(E_T(\mu))$ form a discrete approximation $A(\text{Range}(E(\mu)), \prod_T \text{Range}(E_T(\mu)), r_T)$ for $\text{Range}(E(\mu))$. (see Grigorieff (1975)).

In order to make sure that $\text{rank}(E_T(\mu)) = \text{rank}(E(\mu))$ for T sufficiently large it is sufficient to show that the sequence $E_T(\mu)$ is discretely compact (see Stummel (1971)) because $E(\mu)$ has finite rank.

We recall that the sequence of bounded operators A_{T_n} in $C([1,T_n])$ is discretely compact if for every bounded sequence $f_{T_n} \varepsilon C([1,T_n])$ there is a subsequence $f_{T_{n_k}}$ so that $A_{T_{n_k}} f_{T_{n_k}}$ is convergent to an element in $C_0([1,\infty])$.

We write

$$E_T r_T = r_T E + (E_T r_T - r_T E) : C_0([1,\infty]) \to C([1,T]) \; . \tag{3.26}$$

Obviously

$$E_T r_T - r_T E = \frac{1}{2\pi i} \int_\Gamma (z - V_T)^{-1} (V_T r_r - r_T V)(z - V)^{-1} dz \tag{3.27}$$

holds.

For an arbitrary $f \varepsilon C_0([1,\infty])$ the function $e_T = (V_T r_T - r_T V) f \varepsilon C([1,T])$ is the solution of the problem

$$e_T' - t^\alpha A(t) e_T = 0 \; , \quad 1 < t < T \tag{3.28}$$

$$B e_T(1) = 0 \tag{3.29}$$

$$S(T) e_T(T) = -S(T)(Vf)(T) \; . \tag{3.30}$$

Proceeding similarly to de Hoog and Weiss (1980b) we can express e_T explicitely.

We substitute $F\tilde{e}_T = e_T$ where F is as in (2.1) and get the problem

$$\tilde{e}_T' = t^\alpha J(\infty)\tilde{e}_T + t^\alpha (J(t) - J(\infty))\tilde{e}_T \tag{3.31}$$

$J(t)$, $J(\infty)$ are as in (2.11), (2.2). Now we write

$$\tilde{e}_T = \tilde{e}_T^+ \xi_1^T + \tilde{e}_T^- \xi_2^T \; , \quad \xi_1^T \varepsilon C^{r_+} \; , \quad \xi_2^T \varepsilon C^{r_-} \tag{3.32}$$

where $\tilde{e}_T^+$, $\tilde{e}_T^-$ fulfill

$$\tilde{e}_T^+(t) = \begin{bmatrix} \exp\left(\frac{J_\infty^+}{\alpha+1}(t^{\alpha+1} - T^{\alpha+1})\right) \\ \\ 0 \end{bmatrix} + (H_T(J - J(\infty))\tilde{e}_T^+)(t) \tag{3.33}$$

$$\tilde{e}_T^-(t) = \begin{bmatrix} 0 \\ \\ \exp\left(\frac{J_\infty^-}{\alpha+1} t^{\alpha+1}\right) \end{bmatrix} + (H_T(J - J(\infty))\tilde{e}_T^-)(t) \tag{3.34}$$

where H_T is a suitable solution operator of the problem

$$z' = t^\alpha J(t)z + t^\alpha g(t)\ , \quad 1 \le t \le T\ , \quad g \in C([1,T])\ . \tag{3.35}$$

We choose

$$(H_T g)(t) = (H\hat{g})(t), \quad 1 \le t \le T \tag{3.36}$$

with H defined in (2.4) where $\hat{g}$ has been set to

$$\hat{g}(t) = \begin{cases} g(t)\ , & 1 \le t \le T \\ g(T)\ , & T \le t \le \infty \end{cases}\ . \tag{3.37}$$

Because H is bounded on $[\delta,\infty]$ independently of δ we get

$$\|H_T(J - J(\infty))\|_{[\delta,T]} \le \text{const.}\ \|J - J(\infty)\|_{[\delta,T]} \le \tfrac{1}{2} \tag{3.38}$$

for δ, T sufficiently large. The operator

$$I - H_T(J - J(\infty)) : C([\delta,T]) \to C([\delta,T]) \tag{3.39}$$

is invertible and $\tilde{e}_T^+, \tilde{e}_T^- \in C([\delta,T])$ are uniquely defined and can be continued to $[1,T]$.

Inserting (3.32) into the boundary conditions (3.29), (3.30) gives:

$$\begin{bmatrix} BF\tilde{e}_T^+(1) & BF\tilde{e}_T^-(1) \\ S(T)F\tilde{e}_T^+(T) & S(T)F\tilde{e}_T^-(T) \end{bmatrix} \begin{pmatrix} \xi_1^T \\ \xi_2^T \end{pmatrix} = \begin{pmatrix} 0 \\ -S(T)(Vf)(T) \end{pmatrix}\ . \tag{3.40}$$

de Hoog and Weiss (1980a) have shown that

$$\text{(a)}\ \lim_{T\to\infty} \tilde{e}_T^+(T) = G_+\ , \quad \text{(b)}\ \|\tilde{e}_T^- - r_T\psi_-\|_{[1,T]} \to 0 \quad \text{as} \quad T \to \infty \tag{3.41}$$

hold, where ψ_- is defined in (2.12).

A block system of the form

$$\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{3.42}$$

where $\bar{B}, \bar{C}$ are quadratic matrices is uniquely soluble if and only if $\bar{B}, (\bar{C} - \bar{D}\bar{B}^{-1}\bar{A})$ are invertible and the solution is

(3.43)

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{bmatrix} -(\bar{C} - \bar{D}\bar{B}^{-1}\bar{A})^{-1}\bar{D}\bar{B}^{-1} & (\bar{C} - \bar{D}\bar{B}^{-1}\bar{A})^{-1} \\ \bar{B}^{-1} + \bar{B}^{-1}\bar{A}(\bar{C} - \bar{D}\bar{B}^{-1}\bar{A})^{-1}\bar{D}\bar{B}^{-1} & -\bar{B}^{-1}\bar{A}(\bar{C} - \bar{D}\bar{B}^{-1}\bar{A})^{-1} \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix},$$

The off diagonal matrices in (3.40) are invertible, their inverses are bounded as $T \to \infty$, the matrix in the (2,2) position converges to 0 as $T \to \infty$ and the matrix in the (1,1) position is bounded, and therefore the system is invertible for T sufficiently large. Moreover

(3.44) $$\lim_{T\to\infty} \tilde{e}_T^{+}(1) = 0$$

holds because we get from the series expansion of (3.33):

$$\|\tilde{e}_T^{+} - [\begin{matrix} \exp(\frac{J_\infty^+}{\alpha+1}(w-T^{\alpha+1})) \\ 0 \end{matrix}]\|_{[\delta,T]} \le C\|(J-J(\infty))[\begin{matrix} \exp(\frac{J_\infty^+}{\alpha+1}(w-T^{\alpha+1})) \\ 0 \end{matrix}]\|_{[\delta,T]}$$

where $w(t) = t^{\alpha+1}$ has been set. The right hand side of this inequality can be estimated by

(3.44)(b) $$\bar{\sigma}(\kappa,T) = C \max_{i=1(1)k} \max_{t\in[\delta,T]} (\|J(t)-J(\infty)\|\exp(\frac{\kappa}{\alpha+1}(t^{\alpha+1}-T^{\alpha+1}))(T^{\alpha+1}-t^{\alpha+1})^i)$$

where κ is the smallest (positive) real part of the eigenvalues of J_∞^+ and k is the dimension of the largest Jordan block with eigenvalue with real part κ. Obviously $\bar{\sigma}(\kappa,T) \to 0$ as $T \to \infty$ and (3.44) follows by continuation to $[1,T]$.

We get from (3.40), (3.41), (3.43), (3.44)

(3.45)(a) $$\xi_1^T = -((S(T)FG_+)^{-1} + o(T))S(T)(Vf)(T)$$

(3.45)(b) $$\xi_2^T = o(T)S(T)(Vf)(T)$$

holds. For $g \in C_0([1,\infty])$ we therefore get:

(3.45)(c) $$(V_T r_T - r_T V)(z-V)^{-1}g = (-F\tilde{e}^{+}(S_T(T)FG)_+^{-1} + F\tilde{e}^{-}o_T(T) + o(T))S(T)(V(z-V)^{-1}g)(T).$$

Obviously $h = V(z - V)^{-1}g$ is the solution to the problem

(3.46) $$h' - t^\alpha(A(t) + \frac{1}{z}G(t))h = \frac{1}{z}t^\alpha G(t)g(t)$$

(3.47) $$Bh(1) = 0$$

(3.48) $$h \in C_0([1,\infty])$$

We define

$$(3.49) \qquad A(t,\lambda) = A(t) + \lambda G(t) \quad \text{for} \quad t \in [1,\infty]$$

and the family of operators $\hat{H}(\lambda) : C([\delta,\infty]) \to C([\delta,\infty])$

$$(3.50) \quad (\hat{H}(\lambda)g)(t) = \hat{\phi}(t,\lambda) \int_{\infty}^{t} P_+(\lambda)\hat{\phi}^{-1}(s,\lambda)s^{\alpha}g(s)ds + \hat{\phi}(t,\lambda) \int_{\delta}^{t} P_-(\lambda)\hat{\phi}^{-1}(s,\lambda)s^{\alpha}g(s)ds,$$

for $\delta > 1$ where

$$(3.51) \qquad \hat{\phi}(t,\lambda) = \exp(A(\infty,\lambda)\,\frac{t^{\alpha+1}}{\alpha+1})$$

$$(3.52) \qquad (a) \quad P_+(\lambda) = F(\lambda)D_+F^{-1}(\lambda), \qquad (b) \quad P_-(\lambda) = F(\lambda)D_-F^{-1}\lambda)$$

hold. Obviously $\hat{H}(0) = FHF^{-1}$ for H as in (2.4). The projections $P_+(\lambda)$, $P_-(\lambda)$ are holomorphic in compact sets $K \subset \Omega$ (See Kato (1966)).

Using the techniques of de Hoog and Weiss (1980a) and Markowich (1980b) we conclude that

$$(3.53) \qquad \|\hat{H}(\lambda)\|_{[\delta,\infty]} \leq C(\lambda)$$

where $C(\lambda)$ is independent of δ and bounded on compact set $K \subset \Omega$. Moreover it is an easy exercise to show that $(H(\lambda)g(\cdot,\lambda))(t)$ is holomorphic in $\mathring{K}$ for all $t \in [\delta,\infty]$ if $g(t,\lambda)$ is continuous for $t \in [1,\infty]$ and holomorphic for all t in $\lambda \in K$.

Proceeding as in Chapter 2 we rewrite (3.46)

$$(3.54) \qquad h' = t^{\alpha}A(\infty,\tfrac{1}{z})h + t^{\alpha}\underbrace{(A(t,\tfrac{1}{z}) - A(\infty,\tfrac{1}{z}))}_{B(t,\frac{1}{z})}h + \tfrac{1}{z}t^{\alpha}G(t)g(t)$$

and get

$$(3.55) \qquad h(t) = \hat{\phi}(t,\tfrac{1}{z})W_-(\tfrac{1}{z})\zeta + (\hat{H}(\tfrac{1}{z})B(\cdot,\tfrac{1}{z})h)(t) + \tfrac{1}{z}(\hat{H}(\tfrac{1}{z})Gg)(t)$$

where the columns of the holomorphic $n \times r_-$ matrix $W_-(\frac{1}{z})$ span the range of $P_-(\frac{1}{z})$ and $\zeta \in C^{r_-}$. Because of (3.53) there is a fixed δ so that

$$(3.56) \qquad \|\hat{H}(\tfrac{1}{z})B(\cdot,\tfrac{1}{z})\|_{[\delta,\infty]} \leq 1/2 \quad \text{for all} \quad z \in \bar{S}_{\mu}$$

where $\bar{S}_{\mu}$ is the closed disk contoured by Γ.

Setting

(3.57)(a) $$\hat{\psi}_-(\cdot,\lambda) = (I - \hat{H}(\lambda)B(\cdot,\lambda))^{-1}\hat{\phi}(\cdot,\lambda)W_-(\lambda)$$

(3.57)(b) $$\hat{\psi}(Gf,\lambda) = (I - \hat{H}(\lambda)B(\cdot,\lambda))^{-1}\hat{H}(\lambda)Gg \ .$$

The general solution of (3.46), (3.48) on $[\delta,\infty]$ is

(3.58) $$h(t) = \hat{\psi}_-(t,\tfrac{1}{z})\zeta + \frac{1}{z}\,\hat{\psi}(Gg,\tfrac{1}{z})(t)$$

$\hat{\psi}_-,\hat{\psi}(Gg,\lambda)$ can be uniquely extended to $[1,\infty]$.

Proceeding as in (2.19) using (3.53) we get

(3.59)(a) $$\|\hat{\psi}_-(T,\tfrac{1}{z})\| = o(T) \text{ uniformly for } z \in \Gamma$$

and

(3.59)(b) $$\|\zeta\| = O(\|g\|_{[1,\infty]}) \text{ uniformly for } z \in \Gamma$$

can be concluded as in Chapter 2 using (3.56).

(3.56) and the uniform convergence of the series expansion of (3.57) assures the analyticity of $\hat{\psi}(Gg,\frac{1}{z})(T)$ for $z \in \bar{S}_\mu$. Therefore

(3.60) $$(V_T r_T - r_T V)(z-V)^{-1}g = (-F\tilde{e}_T^+(S(T)FG_+)^{-1} + F\tilde{e}_T^- \, o(T) + \\ + o(T))(\tfrac{1}{z}S(T)\hat{\psi}(Gg,\tfrac{1}{z})(T) + o(T)) \ .$$

$u_T = (z - V_T)^{-1}F\tilde{e}_T^+$ is the solution to the problems

(3.61)(a) $$u_T' - t^\alpha A(t,\tfrac{1}{z})u_T = 0$$

(3.61)(b) $$Bu_T(1) = \tfrac{1}{z}BF\tilde{e}_T^+(1)$$

(3.61)(c) $$S(T)u_T(T) = \tfrac{1}{z}S(T)F\tilde{e}_T^+(T) \ .$$

We set similarly to (3.32)

(3.62)(a) $$u_T(t) = e_T^+(t,\tfrac{1}{z})\zeta_1^T + e_T^-(t,\tfrac{1}{z})\zeta_2^T$$

where

(3.62)(b) $$e_T^+(t,\lambda) = \hat{\phi}(t,\lambda)\hat{\phi}^{-1}(T,\lambda)W_+(\lambda) + (\hat{H}(\lambda,T)B(\cdot,\lambda)e_T^+(\cdot,\lambda))(t)$$

(3.62)(c) $$e_T^-(t,\lambda) = \hat{\phi}(t,\lambda)W_-(\lambda) + (\hat{H}(\lambda,T)B(\cdot,\lambda)e_T^-(\cdot,\lambda))(t)$$

hold. The columns of the holomorphic matrix $W_+(\lambda)$ span the range of $P_+(\lambda)$ and

$\hat{H}(\lambda,T): C([\delta,T]) \to C([\delta,T])$ so that

$$(3.62)(d) \qquad \hat{H}(\lambda,T)g := \hat{H}(\lambda)\hat{g}\,, \quad \hat{g}(t) = \begin{cases} g(t)\,, & \delta \le t \le T \\ g(T), & t > T \end{cases}$$

holds. Because of (3.53) the equations (3.62)(b), (c) are uniquely soluble for all $z \in \bar{S}_\mu$ if δ is sufficiently large and the analyticity of $e_T^{\pm}(t,\frac{1}{z})$ for all $t \in [1,T]$ follows by the above argument and by continuation from $[\delta,T]$ to $[1,T]$.

Moreover we derive as in (3.41) from de Hoog and Weiss (1980a) (3.63)

$$\text{(a)} \quad \lim_{T\to\infty} e_T^{+}(T,\tfrac{1}{z}) = W_+(\tfrac{1}{z}), \qquad \text{(b)} \quad \| e_T^{-}(\cdot,\tfrac{1}{z}) - r_T\hat{\psi}_-(\cdot,\tfrac{1}{z})\|_{[1,T]} \to 0$$

and as in (3.44)

$$(3.63)(a) \qquad \lim_{T\to\infty} e_T^{+}(1,\tfrac{1}{z}) = 0$$

uniformly for $z \in \Gamma$.

Inserting into the boundary condition (3.61)(b),(c) results in a block system of the form (3.42) and using (3.43) gives

$$(3.64)(a) \qquad u_T = e_T^{+}(\cdot,\tfrac{1}{z})(-\tfrac{1}{z}(S(T)W_+(\tfrac{1}{z}))^{-1}S(T)FG_+ + o(T)) + e_T^{-}(\cdot,\tfrac{1}{z})o(T)$$

uniformly for $z \in \Gamma$. The solvability of the problem (3.61) follows from the invertibility of the (analytic) matrix $S(T)W_+(\frac{1}{z})$ which is a direct consequence of Assumption (3.20). Putting all together gives

$$(3.64)(b) \qquad \begin{aligned} &((z - V_T)^{-1}(V_T r_T - r_T V)(z - V)^{-1}g)(t) = \\ &= \frac{1}{z^2}\, e_T^{+}(t,\tfrac{1}{z})(S(T)W_+(\tfrac{1}{z}))^{-1}S(T)\hat{\psi}(Gg,\tfrac{1}{z})(T) + \\ &+ e_T^{-}(t,\tfrac{1}{z})A_1(T,z) + (z-V_T)^{-1}F\tilde{e}_T^{-}(t)A_2(T,z) + A_3(T,z) \end{aligned}$$

where

$$\|A_i(t,z)\| \le o(T)\|g\|_{[1,\infty]} \qquad \text{for} \quad i = 1,2,3$$

uniformly for $z \in \Gamma$.

The first summand on the right hand side of (3.64)(b) is holomorphic in $\bar{S}_\mu$ and therefore its contour integral along Γ vanishes.

Now we define the imbedding operators

$$(3.66)\qquad (a)\quad i_T : C([1,T]) \to C_0([1,\infty]),\ (b)\quad (i_T f_T)(t) = \begin{cases} f_T(t) & 1 \le t \le T \\ f_T(T)\,\frac{T}{t}, & t > T\ . \end{cases}$$

Obviously $r_T i_T f_T = f_T$ and $\|i_T f_T\|_{[1,\infty]} = \|f_T\|_{[1,T]}$ holds.

Because of (3.64), (3.65) the operator

$$(3.67)\qquad (z - V_T)^{-1}(V_T r_T - r_T V)(z - V)^{-1} i_T \quad C([1,T]) \to C([1,T])$$

is discretely compact for every $z \in \Gamma$ and the countour- integral-operator (3.27) is also discretely compact (see Grigorieff (1975). Because Range(E) is finite dimensional $r_T E i_T$ is discretely compact and so is $E_T r_T i_T = E_T$ and

$$(3.68)\qquad \operatorname{rank}(E_T(\mu)) = \operatorname{rank}(E(\mu))$$

for T sufficiently large.

Therefore it is guaranteed that the eigenvalue $\mu = \frac{1}{\lambda}$ is stable with regard to the V_T's (see Grigorieff (1975)) so that there are exactly $m = \operatorname{rank}(E(\mu))$ eigenvalues $\mu_T^1,\dots,\mu_T^m$ of V_T which converge to μ and the estimates

$$(3.69)\qquad \operatorname{gap}(\operatorname{Range}(E_T), r_T(\operatorname{Range}(E))) \le \text{const.}\|(V_T r_T - r_T V)|_{\operatorname{Range}(E)}\|_{[1,T]}$$

and

$$(3.70)\qquad \max\left(\left|\frac{1}{\hat{\mu}_T} - \lambda\right|,\ \max_i |\lambda_T^i - \lambda|^{\beta}\right) \le \text{const.}\|(V_T r_T - r_T V)|_{\operatorname{Range}(E)}\|_{[1,T]}$$

hold (see Grigorieff (1975)).

Under the assumption

$$(3.71)\qquad A\left(\frac{1}{\cdot}\right),\ G\left(\frac{1}{\cdot}\right) \in C^{(\alpha+1)k_-(\lambda)+1}\left([0,\frac{1}{\delta}]\right),\quad \delta > 1$$

where $k_-(\lambda)$ is largest algebraic multiplicity of an eigenvalue $\nu(\lambda)$ of the matrix $A(\infty) + \lambda G(\infty)$ with a negative real part, the estimates (2.58), (2.59), where $\nu_- = \nu_-(\lambda)$ is now the largest negative real part of the eigenvalues of $A(\infty) + \lambda G(\infty)$, follow.

However, a stronger estimate can be derived by proceeding as Osborn (1975) did but without carrying out the last estimates which lead to his Theorems 1,2,3. In the same way the estimates given by Grigorieff (1975) can be changed. We get

(3.72)(a) $$\operatorname{gap}(\operatorname{Range}(E_T),\ r_T(\operatorname{Range}(E))) \leq \text{const.}\|(E_T r_T - r_T E)|_{\operatorname{Range}(E)}\|$$

(3.72)(b) $$\max\left(\left|\frac{1}{\hat{\mu}_T} - \lambda\right|,\ \max_{i=1(1)m} |\lambda_T^i - \lambda|^{\beta}\right) \leq \text{const.}\|(E_T(V_T r_T - r_T V))|_{\operatorname{Range}(E)}\|.$$

An estimate for the right hand side of (3.72)(a) can be obtained by using (3.64)(b) with $g \in \operatorname{Range}(E)$ and (3.65):

(3.73)(a) $$\operatorname{gap}(\operatorname{Range}(E_T),\ r_T(\operatorname{Range}(E))) \leq \text{const.}o(T)\exp\left(\frac{(\nu_-(\lambda)+\varepsilon)}{\alpha+1} T^{\alpha+1}\right).$$

In a similar way we get

(3.73)(b) $$\max\left(\left|\frac{1}{\hat{\mu}_T} - \lambda\right|,\ \max_i|\lambda_T^i - \lambda|^{\beta}\right) \leq \text{const. } o(T)\exp\left(\frac{(\nu_-(\lambda)+\varepsilon)}{\alpha+1} T^{\alpha+1}\right).$$

Therefore the standard error estimates are not sharp for the whole class of problems.

Retracing the history of the $o(T)$ in (3.73) we get

(3.74) $$o(T) \leq \max(\|A(T) - A(\infty)\|, \|G(T) - G(\infty)\|,\ \bar{\sigma}(\kappa - \varepsilon, T))$$

where $\bar{\sigma}$ is defined in (3.44)(b) and ε is small when the radius of Γ is sufficiently small.

$S(T)$ can be chosen independently of λ for a large class of problems, for example if $G(\infty)$ is regular. In this case we can set $G(\infty) = I$ because this always can be achieved by a linear transformation. Then $F(\lambda) \equiv F$ and (3.20) is equal to (2.41). Ω is then the strip $\nu_- < \lambda < \nu_+$ where ν_- is the largest negative and ν_+ is the smallest positive real part of eigenvalues of $A(\infty)$. In this case the asymptotic boundary condition (2.60) can be used.

In the case that $G(\infty) \neq 0$ is not regular $S(T)$ can be chosen independently of λ if we know a sufficiently close approximation $\tilde{\lambda} \in \Omega$ to an eigenvalue λ of (1.1), (1.2), (1.3). Then we rewrite (1.1) as

(3.75) $$y' - t^{\alpha}(A(t) + \tilde{\lambda}G(t))y = \gamma t^{\alpha}G(t)y, \quad \gamma = \lambda - \tilde{\lambda}.$$

$$\tilde{A}(t) = A(t) + \tilde{\lambda}G(t)$$

The isolatedness of the eigenvalues $\lambda \in \Omega$ guarantees that the problem

(3.76)(a) $$y_h' - t^{\alpha}\tilde{A}(t)y_h = 0$$

(3.76)(b) $$By_h(1) = 0$$

(3.76)(c) $$y_h \in C([1,\infty])$$

has only the trivial solution $y_h \equiv 0$ if $\tilde{\lambda}$ is sufficiently close to λ. Therefore the above theory can be used for the eigenvalue problem (3.75), (1.2), (1.3) (with γ as eigenparameter). The change consists of taking $\tilde{\lambda}$ instead of 0 as reference point.

We set

$$S \equiv S(T) = (G_+)^T F^{-1}(\tilde{\lambda}) \tag{3.77}$$

where the superscript T denotes transposition and $F(\tilde{\lambda})$ transforms $A(\infty) + \tilde{\lambda}G(\infty) = \tilde{A}(\infty)$ to its Jordan canonical form.

Defining $F = F(\tilde{\lambda})$ (2.40), (2.41) follows immediately.

From the analysis for the approximating problems (1.4), (1.5), (1.6) it follows that it is sufficient to require that (3.20) holds locally if the particular eigenvalue λ is to be calculated. "Locally" means in this context is the closed set bounded by the contour $\frac{1}{\mu}(\Gamma)$ defined in (3.9).

Since the family of projection $F(\lambda)D_+F^{-1}(\lambda)$ is holomorphic in $K \subset \Omega$, K compact there is a nonsingular $r_+ \times r_+$ matrix $T(\lambda)$ so that

$$W_+(\lambda) = F(\lambda)G_+T(\lambda) \text{ is holomorphic in } \Omega . \tag{3.78}$$

Therefore

$$SF(\lambda)G_+T(\lambda) = (G_+)^T F^{-1}(\tilde{\lambda})(F(\tilde{\lambda})G_+T(\tilde{\lambda}) + O(|\lambda - \tilde{\lambda}|)) = T(\tilde{\lambda}) + O(|\lambda - \tilde{\lambda}|)$$

holds and

$$(SF(\lambda)G_+)^{-1} = T(\lambda)T(\tilde{\lambda})^{-1} + O(|\lambda - \tilde{\lambda}|)) \tag{3.79}$$

for λ sufficiently close to $\tilde{\lambda}$. So (3.20) holds locally for λ close to $\tilde{\lambda}$ and the asymptotic boundary condition (3.77) can be used for the calculation of λ if the initial guess $\tilde{\lambda}$ is sufficiently close to λ.

This analysis leads to the idea to use asymptotic boundary condition which depend on the eigenparameter λ. This leads even in the case that the 'infinite' problem is a linear eigenvalue problem, to nonlinear approximating 'finite' eigenvalue problems which, suggested by Keller (1976) have been successfully used in computation (see Ng and Reid (1980)), and their analysis will be presented in a subsequent paper.

However for many important fluid-dynamical problems it is possible to choose simpler asymptotic boundary conditions. An example is presented in Chapter 5.

4. Imaginary Eigenvalues of $A(\infty)$.

We are now going to neglect the crucial restriction that all eigenvalues of $A(\infty)$ have a non-zero real part but we will require a sufficiently fast convergence of $G(t)$ to 0 which puts us back into the compactness argument of Chapter 2.

We assume that

$$(4.1) \qquad A\left(\frac{1}{\cdot}\right) \in C^{(\alpha+1)\max(k_0,k_-)+1}\left([0,\frac{1}{\delta}]\right), \quad \delta > 1$$

$$(4.2) \qquad \|G(t)\| \le \text{const. } t^{-(\alpha+1)k_0-\varepsilon}, \quad \varepsilon > 0$$

where k_0 is the largest algebraic multiplicity of an eigenvalue of $A(\infty)$ with real part zero and k_- is defined as in Chapter 2 .

Markowich (1980a,b) has shown that there is a solution operator $\bar{H}$ of the inhomogenous problem (2.8) which fulfills

$$(4.3) \qquad \|(\bar{H}f)(t)\| \le \text{const. } t^{-\varepsilon}(\ell n t)^j \|f\|_{[1,\infty]}, \quad 1 \le j \le n$$

if (4.2) holds.

Therefore, if the homogenous problems (2.8), (2.9), (2.10) has only the trivial solution $y \equiv 0$ then the operator V (see (2.16)) is well defined and as the sum of a degenerate and a compact opeprator it is compact and the same consideration as in Chapter 2 hold for the eigenvalues and the generalized eigenvectors except the decay statements because the eigenvalues with real part zero may produce solutions which are asymptotically constant or which decay algebraically. An algorithm which determines the nature of the basic solutions under the assumption (4.1) is given in Markowich (1980b), Chapters 3 and 4.

The construction of the supplementary boundary condition $S(T)x_T(T) = 0$ for the approximating problems (1.4), (1.5), (1.6) now relies heavily on the asymptotic nature of the basic solutions and is explained in Markowich (1980c), Chapters 3 and 4. The matrix $S(T)$ constructed in the mentioned paper takes care that the basic solutions which are in $C([1,\infty])$ but which do not decay sufficiently fast, are dampened by the multiplication with $S(T)$ so that norm convergence of the operators V_T defined as in (2.35) to V results (see Markowich (1980c), Chapter 4).

Exponential convergence of eigenvalues and spectral subspaces holds if all (basic) solutions of the problem:

$$y' - t^{\alpha}A(t)y = 0 \tag{4.3}$$

$$y \in C([1,\infty]) \tag{4.4}$$

decay exponentially.

5. The Orr-Sommerfeld Equation.

The Orr-Sommerfeld equation (see Ng and Reid (1980)) governs the stability of laminar boundary layers in the parallel flow approximation

(5.1) $$\frac{1}{iR\alpha}\left(\frac{d^2}{dz^2}-\alpha^2\right)^2\phi-\left[(U(z)-\lambda)\left(\frac{d^2}{dz^2}-\alpha^2\right)\phi-U''(z)\phi\right]=0$$

$\alpha \in R$, $\alpha > 0$. $\phi(z)e^{i\alpha(x-\lambda t)}$ is the disturbance stream function, $R > 0$ is the Reynolds number and $U(z)$ is the velocity distribution fulfilling

(5.2) $$U \in C([0,\infty]),\quad U(\infty) = 1,\quad U''(\infty) = 0 .$$

The boundary conditions for the Orr-Sommerfeld problem at $z = 0$ and $z = \infty$ are

(5.3) $$\phi(0) = \phi'(0) = 0$$

(5.4) $$\phi(\infty) = \phi'(\infty) = 0 .$$

This problem is of singular perturbation type for R large, but we disregard that and just derive appropriate asymptotic boundary conditions.

We substitute

(5.5) $$y = (\phi,\phi',\phi'',\phi''')^T$$

and get the problem

(5.6) $$y' = \overbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ f_1(z) & 0 & f_2(z) & 0 \end{bmatrix}}^{A(z)} y = \lambda \overbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & b & 0 \end{bmatrix}}^{G} y,\quad 0 < z < \infty$$

(5.7) $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} y(0) = 0$$

(5.8) $$y \in C([0,\infty])$$

where

(5.9)

(a) $$f_1(z) = -(\alpha^4 + i\alpha R(\alpha^2 U(z) + U''(z)))$$

(b) $$f_2(z) = 2\alpha^2 + i\alpha R U(z)$$

and

$$(5.10) \qquad (a) \quad a = i\alpha^3 R \qquad (b) \quad b = -i\alpha R .$$

The eigenvalues of $A(\infty)$ are

$$(5.11) \qquad \nu_1 = \alpha , \quad \nu_2 = (\alpha^2 + i\alpha R)^{1/2}, \quad \nu_3 = -\alpha , \quad \nu_4 = -(\alpha^2 + i\alpha R)^{1/2}$$

so that $\mathrm{Re}\ \nu_1$, $\mathrm{Re}\ \nu_2 > 0$; $\mathrm{Re}\ \nu_3$, $\mathrm{Re}\ \nu_4 < 0$ and the eigenvalues of $A(\infty) + \lambda G$ are

$$(5.12) \quad \nu_1(\lambda) = \alpha, \ \nu_2(\lambda) = (\alpha^2 + i\alpha R(1-\lambda))^{1/2}, \ \nu_3(\lambda) = -\alpha, \ \nu_4(\lambda) = -(\alpha^2 + i\alpha R(1-\lambda))^{1/2}$$

so that $\mathrm{Re}\ \nu_1(\lambda)$, $\mathrm{Re}\ \nu_2(\lambda) > 0$; $\mathrm{Re}\ \nu_3(\lambda)$, $\mathrm{Re}\ \nu_4(\lambda) < 0$ for all $\lambda \in \mathbb{C}$ holds. Therefore the set Ω defined in Chapter 3 is the whole complex plane. We get

$$(5.13)(a) \qquad J(\infty) = \mathrm{diag}(\nu_1, \nu_2, \nu_3, \nu_4) \quad \text{for } \lambda \neq 1$$

and

$$(5.13)(b) \qquad J(\infty) = \begin{bmatrix} \nu_1 & 1 & & \\ & \nu_1 & 0 & \\ 0 & & \nu_2 & 1 \\ & & & \nu_2 \end{bmatrix} \quad \text{for } \lambda = 1$$

so that

$$(5.14) \qquad G_- = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} , \quad G_+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

holds. We calculate

$$(5.15)(a) \qquad F(\lambda) = \begin{bmatrix} 1 \\ \nu_i(\lambda) \\ \nu_i^2(\lambda) \\ \nu_i^3(\lambda) \end{bmatrix}_{i = 1(1)4} \qquad \text{for } \lambda \neq 1 , \quad F(0) = F$$

$$(5.15) \qquad F(1) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ \alpha & 1 & -\alpha & 1 \\ \alpha^2 & 2\alpha & \alpha^2 & 0 \\ \alpha^3 & 3\alpha^2 & -\alpha^3 & \alpha^2 \end{bmatrix}$$

All eigenvalues λ are isolated and have finite algebraic multiplicities.

We choose the matrix $S(Z)$ which sets up the asymptotic boundary condition $S(Z)y_Z(Z) = 0$, $Z > 0$, independent of Z so that

$$S(Z) \equiv S = (S_{ij})_{\substack{i = 1,2 \\ j = 1,2,3,4}} \tag{5.16}$$

holds. Then the regularity condition (3.20) reduces to:

$$\det \begin{bmatrix} \sum_{j=1}^{4} s_{1j}\alpha^{j-1} & \sum_{j=1}^{4} s_{1j}\nu_2(\lambda)^{j-1} \\ \sum_{j=1}^{4} s_{2j}\alpha^{j-1} & \sum_{j=1}^{4} s_{2j}\nu_2(\lambda)^{j-1} \end{bmatrix} \neq 0 \quad \text{for } \lambda \in C - \{1\} \tag{5.17}$$

and

$$\det \begin{bmatrix} \sum_{j=1}^{4} s_{1j}\alpha^{j-1} & \sum_{j=1}^{4} s_{1j}(j-1)\alpha^{j-2} \\ \sum_{j=1}^{4} s_{2j}\alpha^{j-1} & \sum_{j=1}^{4} s_{2j}(j-1)\alpha^{j-2} \end{bmatrix} \neq 0 \quad \text{for } \lambda = 1 . \tag{5.18}$$

For example the 'natural' asymptotic boundary condition

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{5.19}$$

fulfills (5.17) and (5.18).

The order of convergence for eigenvalues and spectral subspaces of the approximating problems (1.4), (1.5), (1.6), where S fulfills (5.17), (5.18), at a particular eigenvalue λ of (5.6), (5.7), (5.8), can be estimated by

$$o(Z)\exp((\max(-\alpha, \mathrm{Re}\ \nu_4(\lambda)) + \varepsilon)Z/\beta) \tag{5.20}$$

where

$$\mathrm{Re}\ \lambda_4(\lambda) = -\left(\frac{\alpha^2 + \alpha R \mathrm{Im}\lambda}{2} + \left(\frac{(\alpha^2 + \alpha R \mathrm{Im}\lambda)^2}{4} + \frac{\alpha^2 R^2 (1 - \mathrm{Re}\lambda)^2}{4}\right)^{1/2}\right)^{1/2} \tag{5.21}$$

holds and β is the ascent of λ .

Computation of the Orr-Sommerfeld problem using the boundary conditions set up by (5.19) can be found in Grosch and Orsag (1977). They used the Blasius velocity profile $u(z) = 1 + O(e^{-wz^2}), w > 0$.

Their numerical experiments indicate that the order of convergence is $e^{-2\alpha Z}$ in the case that $\alpha < |\mathrm{Re}\ \nu_4(\lambda)| < 1$ and λ has (most likely) ascent 1 . Checking our order formula (3.73), (3.79) gives

$$(5.22) \qquad o(Z) < \max(e^{-wZ^2}, \max_{z\varepsilon[\delta,Z]} e^{-wz^2+(\alpha-\varepsilon)(z-Z)}) = \mathrm{const.}e^{(-\alpha+\varepsilon)Z}$$

and the order of convergence the theory predicts is $e^{-2(\alpha-\varepsilon)Z}$ for eigenvalues and spectral subspaces.

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This paper is concerned with eigenvalue problems for boundary value problems of ordinary differential equations posed on an infinite interval. Problems of that kind occur for example in fluid mechanics when the stability of laminar flows is investigated. Characterizations of eigenvalues and spectral subspaces are given and the convergence of approximating problems which are derived by reducing the infinite interval to a finite but large one and by imposing additional boundary conditions at the far end is proved. Exponential convergence is shown for a large class of problems.

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